

Composite-fermion description of rotating Bose gases at low angular momenta

M.N. Korslund and S. Viefers

Department of Physics, University of Oslo, P.O. Box 1048 Blindern, N-0316 Oslo, Norway

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We study the composite fermion construction at and below the single vortex ($L = N$) state of weakly interacting rotating Bose gases, presenting a new method for handling the large number of derivatives typically occurring via the Slater determinant. Remarkably, the CF wave function at $L = N$ becomes asymptotically *exact* in the thermodynamic limit, even though this construction is not, *a priori*, expected to work in the low angular momentum regime. This implies an interesting mathematical identity which may be useful in other contexts.

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INTRODUCTION

Rotating atomic Bose condensates have been subject to ever increasing theoretical and experimental interest in recent years[1]. On the experimental side, substantial progress has been made in increasing the rotation such as to produce ever larger amounts of vortices [2, 3, 4, 5] and, very recently, coming close to the point where the vortex lattice is expected to melt and the system is expected to enter the quantum Hall regime[6]. Theoretical studies have addressed the entire range of angular momenta, from few-vortex states ($L \sim N$ where L is the total angular momentum and N the number of particles), up to the regime $L \sim N^2$ where the system displays a number of close analogies to the fractional quantum Hall effect (FQHE)[7, 8, 9].

A theoretical tool which has been very successful in describing the physics in the quantum Hall regime is the composite fermion (CF) construction[10]. Originally developed in the context of the fractional quantum Hall effect, CF wave functions have, more recently, been applied to the high angular momentum states of weakly interacting Bose gases[9, 11, 12, 13]. By construction, this method has been assumed to be applicable only in the FQH regime, $L \sim N^2$. However, as was pointed out in [11], there were some indications that the CF wave functions might do surprisingly well even at the lowest angular momenta. In this work we thus discuss the CF construction in the regime $0 \leq L \leq N$. In particular, we present a numerical study which shows that the CF wave function for the single vortex ($L = N$) becomes *exact* in the large N limit. One of our main motivations for studying this case in more detail were earlier calculations by Cooper et al [9], which revealed an increase in the overlap as function of N for up to 10 particles. We confirm that this trend continues in a systematic way for higher N , with the deviation of the overlap from unity decreasing as $\sim 1/N$. Since the exact ground state wave function is known analytically in this case [7, 14, 15], this result suggests a non-trivial mathematical identity, valid asymptotically in the limit $N \rightarrow \infty$, which may turn out useful in other contexts. Our solution involves

a new way of handling the large number of derivatives typically occurring in the CF construction when going to low angular momenta. This large number of derivatives has been one of the main obstacles to taking the CF construction to higher particle numbers than, say, 10 [16]. We have computed the overlap of the single vortex CF wave function with the exact one for up to 43 particles, which is a far higher number than could be achieved previously. This was done by taking advantage of the mathematical properties of the determinants involved, which enabled us to rewrite the wave functions in a simpler way, without derivatives, before performing the numerical calculation of the overlaps. A straightforward modification of this method allowed us to study other angular momenta as well, and revealed some peculiar mathematical properties of the CF construction. We would like to emphasize that this is mainly a technical work in the sense that our main emphasis is on this new method and on the mathematical properties of the CF construction. While our study was performed in an angular momentum regime where exact analytical wave functions are known, we hope that our results and methods may turn out useful to study states beyond the single vortex, where this is not the case. For example, the same approach may be used to study the case $L = 2N$, where the system is expected to be close to the transition between a two- and a three-vortex state. Our preliminary findings indicate that the aforementioned mathematical structures may greatly simplify the CF study of this and other few-vortex states.

The paper is organized as follows: We start by reviewing the construction of CF trial wave functions in the context of rotating hard-core bosons, illustrated by some simple examples. We then present evidence that the overlap of the single vortex CF wave function with the exact one converges towards 1 in the large N limit. Since we believe that (modified versions of) our approach may provide a useful tool for studying other CF wave functions with a high number of derivatives, our method of handling the derivatives is explained in some detail. In the following section, we discuss the region $L < N$. The CF wave functions are, in general, not exact here but do

reproduce the exact wave function for $L = 0, 2, 3$ which, again, suggests some interesting mathematical identities. Finally, we summarize our work.

CF DESCRIPTION OF ROTATING BOSE SYSTEMS

Consider a system of N spinless bosons with mass m in a harmonic trap of strength ω , rotating with angular frequency Ω and interacting via a short-range (delta function) potential H_I . In a rotating frame the Hamiltonian can be written as

$$H = \sum_{i=1}^N \left[\frac{\vec{p}_i^2}{2m} + \frac{1}{2}m\omega^2 r_i^2 \right] - \Omega L_z + H_I \quad (1)$$

where L_z denotes the angular momentum around the rotation axis. Reshuffling a few terms, this is straightforwardly rewritten as

$$H = \sum_{i=1}^N \left[\frac{1}{2m} \left(\vec{p}_i - \vec{A} \right)_\parallel^2 + H_{ho}(z_i) \right] + (\omega - \Omega)L_z + H_I \quad (2)$$

with $\vec{A} = m\omega(-y, x)$, $H_{ho}(z)$ denoting the z -part of the harmonic oscillator Hamiltonian and \parallel denoting the planar (x, y) part of the Hamiltonian. This is how the formal link to the quantum Hall system comes about: We see that the planar part of H takes the form of particles moving in an effective "magnetic" field $\vec{B}_{eff} = \nabla \times \vec{A} = 2m\omega\hat{z}$. Now, the interaction is assumed to be weak in the sense that it does not mix different harmonic oscillator levels. We will be interested, for a given total angular momentum, only in the *lowest* many-body states (the "yrast" band). In this limit, the model may be rewritten as a lowest Landau level (LLL) problem in the effective "magnetic" field $B_{eff} = 2m\omega$ (and of course, $n_z = 0$ for the harmonic oscillator in the z -direction). The Hamiltonian then takes the form

$$H = (\omega - \Omega)L + g \sum_{i < j} \delta^2(\mathbf{r}_i - \mathbf{r}_j) \quad (3)$$

($\hbar = 1$) where we now use L to denote the total angular momentum, $L = \sum_i l_i = L_z$.

The single particle states spanning our Hilbert space (the lowest Landau level) are

$$\eta_{0,l} = \frac{1}{\sqrt{2^{l+1}\pi l!}} z^l e^{-\bar{z}z/4} \quad (4)$$

where $z = \sqrt{2m\omega}(x + iy)$ are complex coordinates denoting the particle positions in the plane, and l is the angular momentum of the state. A general bosonic many-body wave function $\psi(z_1, \dots, z_N)$ will then be a homogeneous, symmetric polynomial in the z_i 's, times the exponential factor $\exp(-\sum_i |z_i|^2/4)$ (which will be suppressed throughout this paper for simplicity). The degree

of the polynomial gives the total angular momentum of the state. A special class of such wave functions are the so-called *composite fermion* (CF) wave functions. They were first introduced by Jain[10] and have been extremely successful in describing FQH states, quantum dots in high magnetic fields [17] and, more recently, highly rotational states of Bose condensates[9, 11, 12, 13]. In quantum Hall physics, the main idea of this construction is, roughly speaking, to attach an even number of flux quanta to each electron, thus mapping them into weakly interacting *composite* fermions which can be thought of as moving in a reduced magnetic field. Technically, "attaching a flux quantum" means multiplying the wave function by a Jastrow factor,

$$\prod_{i < j} (z_i - z_j). \quad (5)$$

We see that the Jastrow factor has the effect of keeping the particles apart – it goes to zero if any two coordinates z_i and z_j approach each other. Therefore, it "takes care of" much of the repulsive interaction between the particles. In the simplest approach, the so-called non-interacting composite fermion (NICF) approach, the wave function is then simply constructed as a Slater determinant of (non-interacting) composite fermions in the reduced magnetic field, times an even power of Jastrow factors. In the case of bosons, whose wave function has to be symmetric rather than antisymmetric, one instead absorbs an *odd* power of flux quanta, mapping the bosons to weakly interacting composite fermions. In other words, bosonic trial wave functions with angular momentum L are constructed as non-interacting fermionic wave functions with angular momentum $L - mN(N-1)/2$, multiplied by an *odd* number m of Jastrow factors ($m = 1$ throughout this paper), and projected onto the LLL,

$$\psi_L = \mathcal{P} \left(f_S(z_i, \bar{z}_i) \prod_{i < j} (z_i - z_j)^m \right). \quad (6)$$

Here, f_S denotes a Slater determinant consisting of single-particle wave functions $\eta_{nl}(z, \bar{z}) \propto z^l L_n^l(z\bar{z}/2)$ where n is the (CF) Landau level index ($l \geq -n$) and L_n^l a generalized Laguerre polynomial. The LLL projection \mathcal{P} amounts to the replacement $\bar{z}_i \rightarrow 2\partial/\partial z_i$ in the polynomial part of the wave function – the recipe is to replace all \bar{z} 's with derivatives in the final polynomial, after multiplying out the Slater determinant and the Jastrow factors and moving all \bar{z} 's to the left. It has been shown[10] that with this projection method, the single-particle wave functions in the CF Slater determinant may be written as

$$\eta_{nl} = z^{n+l} \partial^n, \quad l \geq -n \quad (7)$$

with all derivatives acting only to the right. As this method tends to get computationally heavy in numerical

calculations with many particles and a large number of derivatives, slightly different methods of obtaining LLL wavefunctions have been employed in most of the CF literature[10]. These, too, are often referred to as projection. Nevertheless, in this paper, "projection" will refer to the above brute force procedure.

Before moving on to low angular momenta, let us illustrate the method on two simple and well-known examples in the QH regime: First, consider the case $L = N(N-1)$. Taking $m = 1$, the Slater determinant f_S has to contribute an angular momentum $N(N-1)/2$ and is given by putting all CFs into the lowest CF Landau level, from $l = 0$ to $l = N-1$,

$$f_S = \begin{vmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ z_1^2 & z_2^2 & \dots & z_N^2 \\ \dots & \dots & \dots & \dots \\ z_1^{N-1} & z_2^{N-1} & \dots & z_N^{N-1} \end{vmatrix} \equiv \prod_{i < j} (z_i - z_j). \quad (8)$$

We immediately see from Eq.(6) that the full wave function is simply the bosonic Laughlin wave function

$$\psi_L = \prod_{i < j} (z_i - z_j)^2 \quad (9)$$

with angular momentum $L = N(N-1)$. This is the *exact* ground state for the delta function interaction. Next, consider the angular momentum $N(N-1) - N$, corresponding to a bosonic quantum Hall "quasiparticle" (as opposed to quasihole). In order to *decrease* the angular momentum by N as compared to the Laughlin state, we need to move one CF to the second Landau level, *i.e.* construct the Slater determinant

$$f_S = \begin{vmatrix} \bar{z}_1 & \bar{z}_2 & \dots & \bar{z}_N \\ 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ z_1^2 & z_2^2 & \dots & z_N^2 \\ \dots & \dots & \dots & \dots \\ z_1^{N-2} & z_2^{N-2} & \dots & z_N^{N-2} \end{vmatrix}. \quad (10)$$

This gives the full trial wave function (again, apart from the exponential factor)

$$\begin{aligned} \psi_{qp} &= \sum_{i=1}^N (-1)^i \partial_i \prod_{k < l; k, l \neq i} (z_k - z_l) \prod_{m < n} (z_m - z_n) \\ &\propto \sum_{i=1}^N \sum_{j \neq i} \frac{1}{z_i - z_j} \prod_{k \neq i} (z_i - z_k)^{-1} \psi_L \end{aligned} \quad (11)$$

with ψ_L denoting the Laughlin state (9). This wave function has very high overlap with the exact one (*e.g.*, 99.7 % for 4 bosons [11]). Its fermionic counterpart has been proven to capture correctly both the fractional charge and the anyonic statistics of the QH quasielectron[18, 19], and the same is expected to be the case for this bosonic version.

Trial wave functions for other yrast states are constructed in similar ways. The lower the angular momentum, the larger the number of derivatives.

SINGLE VORTEX, $L = N$

In the examples above, we considered angular momenta $L \sim N^2$. Let us now turn to the regime $L \sim N$ which, for large systems, corresponds to much smaller angular momenta. Note that since the Jastrow factor in Eq.(6) has itself an angular momentum of $mN(N-1)/2$, we need to *act with* $\mathcal{O}(N^2)$ derivatives in order to get down to $L \sim N$. One would expect that these derivatives acting on the Jastrow factor destroy most of the good correlations which are at the very heart of the CF construction, which would thus fail in this regime. Surprisingly, however, we shall see that this is not the case. As shown in [11], the Slater determinant for the ground state trial wave function at $L = N$ is constructed by occupying the single-particle states $\eta_{n,-n}$ for $n = N-2$ through $n = 0$, and the state $\eta_{0,1}$,

$$f_S = \begin{vmatrix} z_1 & z_2 & \dots & z_N \\ 1 & 1 & \dots & 1 \\ \bar{z}_1 & \bar{z}_2 & \dots & \bar{z}_N \\ \bar{z}_1^2 & \bar{z}_2^2 & \dots & \bar{z}_N^2 \\ \dots & \dots & \dots & \dots \\ \bar{z}_1^{N-2} & \bar{z}_2^{N-2} & \dots & \bar{z}_N^{N-2} \end{vmatrix} \quad (12)$$

where, again, the \bar{z} :s are to be replaced by derivatives acting on the Jastrow factor. Thus, the trial wave function may be written as [11]

$$\psi_{(L=N)} = \sum_{n=1}^N (-1)^n z_n \prod_{k < l; k, l \neq n} (\partial_k - \partial_l) \prod_{i < j} (z_i - z_j) \quad (13)$$

On the other hand, it has been shown[7, 14, 15] that the *exact* wave function for $L = N$, the single vortex, is given by

$$\psi_{(L=N)}^{ex} = \prod_{i=1}^N (z_i - Z) \quad (14)$$

where $Z = \sum_i z_i / N$ is the center of mass. It was shown in Ref. [9] that, surprisingly, the overlap between (13) and (14) appeared to increase with increasing particle number (but only results up to $N = 10$ were available at the time). Here we have studied this point systematically and computed the overlaps for up to 43 particles. The mathematical approach which enabled us to handle such large numbers of derivatives, will be described shortly. First, however, let us state the result: *The overlap between the CF trial wave function (13) and the exact one (14) converges towards 1 in the large N limit*, see fig. 1. The overlap equals 98.64% already at $N = 5$ and 99.47%

for $N = 10$. For $N = 43$ it equals 99.90%. We have checked that this is not a trivial result in the sense that the two wave functions might simply share the same leading term and all other terms become irrelevant for large N . Rather, we found that the leading term of the two wave functions, $\prod_i z_i$, only corresponds to a weight of about 50% of the total wave function, and that the sub-leading terms of the two wave functions converge towards each other as well. This suggests that the functions (13) and (14) *are identically equal to each other in the limit* $N \rightarrow \infty$. This is a non-trivial mathematical identity which we have so far not been able to prove analytically; however, we noticed by simple curve-fitting, that the deviation of the overlap from unity decreases as $\sim 1/N$. We will discuss other, similar identities in the section on $L < N$.

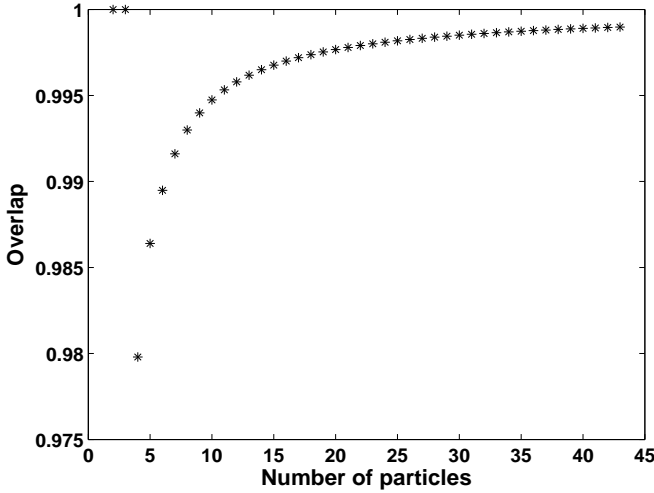


FIG. 1: The overlap between the $L = N$ CF trial wave function (13) and the exact one (14) as function of the number of particles.

Combinatorics and numerical method

For the sake of clarity, we will illustrate our approach to the single vortex wave function for the simple case $N = 4$. The generalization to arbitrary N is straightforward, and the general result will be stated at the end. After projection, the CF Slater determinant (12) takes the form

$$f_S = \begin{vmatrix} z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \\ \partial_1 & \partial_2 & \partial_3 & \partial_4 \\ \partial_1^2 & \partial_2^2 & \partial_3^2 & \partial_4^2 \end{vmatrix}. \quad (15)$$

This determinant may be rewritten as a permutation sum,

$$f_S = \sum_{\pi} \epsilon(\pi) z_{\pi(1)}^1 z_{\pi(2)}^0 \partial_{\pi(3)}^1 \partial_{\pi(4)}^2 \quad (16)$$

with π denoting all permutations over the particle indices (1,2,3,4), and $\epsilon(\pi) = \pm 1$ for even/odd permutations. Similarly, we rewrite the Jastrow factor as

$$J = \begin{vmatrix} 1 & 1 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 \\ z_1^3 & z_2^3 & z_3^3 & z_4^3 \end{vmatrix} = \sum_{\sigma} \epsilon(\sigma) z_{\sigma(1)}^0 z_{\sigma(2)}^1 z_{\sigma(3)}^2 z_{\sigma(4)}^3. \quad (17)$$

Now, the first crucial observation is that an equivalent way of writing the permutation sum (17) is by fixing the particle indices and summing over all permutations of the exponents instead,

$$J = \sum_{\sigma} \epsilon(\sigma) z_1^{\sigma(1)-1} z_2^{\sigma(2)-1} z_3^{\sigma(3)-1} z_4^{\sigma(4)-1}. \quad (18)$$

Moreover, since reordering the terms will leave the sum unchanged, we may apply any permutation π to all the indices as long as we make sure to change the signs accordingly:

$$J = \sum_{\sigma} \epsilon(\pi\sigma) z_{\pi(1)}^{\sigma(1)-1} z_{\pi(2)}^{\sigma(2)-1} z_{\pi(3)}^{\sigma(3)-1} z_{\pi(4)}^{\sigma(4)-1}. \quad (19)$$

Noting that $\epsilon(\pi)\epsilon(\pi\sigma) = \epsilon(\sigma)$, we can now combine the Slater determinant f_S and the Jastrow factor J to give the following expression for the (unnormalized) single vortex wave function,

$$\psi_{(L=N)} = \sum_{\pi} \sum_{\sigma} \epsilon(\sigma) z_{\pi(1)}^1 z_{\pi(2)}^0 \partial_{\pi(3)}^1 \partial_{\pi(4)}^2 \times z_{\pi(1)}^{\sigma(1)-1} z_{\pi(2)}^{\sigma(2)-1} z_{\pi(3)}^{\sigma(3)-1} z_{\pi(4)}^{\sigma(4)-1}. \quad (20)$$

We note that the outer sum is simply a (symmetric) sum over all permutations π of the particles indices. Therefore, it is sufficient to study one term, denoted as f_U , in this permutation sum and perform the symmetrization only at the very end. Choosing this term to be $\pi = (1, 2, 3, 4)$, we get

$$\begin{aligned} f_U &= \sum_{\sigma} \epsilon(\sigma) z_1 \partial_3 \partial_4^2 \times z_1^{\sigma(1)-1} z_2^{\sigma(2)-1} z_3^{\sigma(3)-1} z_4^{\sigma(4)-1} \\ &= \sum_{\sigma} \epsilon(\sigma) z_1^{\sigma(1)} z_2^{\sigma(2)-1} (\sigma(3) - 1) z_3^{\sigma(3)-2} \\ &\quad \times (\sigma(4) - 1)(\sigma(4) - 2) z_4^{\sigma(4)-3}. \end{aligned} \quad (21)$$

The next important observation is that the above sum may again be expressed as a determinant,

$$f_U = \begin{vmatrix} z_1 & 1 & 0 & 0 \\ z_1^2 & z_2 & 1 & 0 \\ z_1^3 & z_2^2 & 2z_3 & 2 \\ z_1^4 & z_2^3 & 3z_3^2 & 6z_4 \end{vmatrix}. \quad (22)$$

Thus, to summarize, we have expressed both the Slater determinant and the Jastrow factor in (6) as permutation sums, combined these, performed all derivatives, and cast

the final expression in the form of a new determinant. The full wave function $\psi_{(L=N)}$ is obtained by adding all permutation of the particle indices in (22). We have obtained an explicit form of the CF polynomial, without derivatives, which may then be used in numerical calculations, to compute overlaps etc.

The generalization of (22) to arbitrary N is straightforward; one finds that

$$f_U = \begin{vmatrix} z_1 & 1 & 0 & \cdots & 0 \\ z_1^2 & z_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ z_1^{N-1} & z_2^{N-2} & \binom{N-2}{1} z_3^{N-3} & \cdots & 1 \\ z_1^N & z_2^{N-1} & \binom{N-1}{1} z_3^{N-2} & \cdots & \binom{N-1}{N-2} z_N \end{vmatrix}. \quad (23)$$

A particularly appealing property of this matrix is its near-triangular form, which greatly reduces the calculation cost.

Before ending this section, let us briefly outline how we used this result to compute overlaps between the CF wave function and the exact one (14). The polynomial part of $\psi_{(L=N)}$ may be regarded as a vector in the space of symmetric polynomials. A convenient basis for this space consists of the symmetrized sums of individual polynomial terms. For example, the symmetrization of $z_1 z_2 z_3^2$ is

$$\mathcal{S}[z_1 z_2 z_3^2] = 2z_1 z_2 z_3^2 + 2z_1 z_2 z_4^2 + 2z_1 z_3 z_2^2 + \dots \quad (24)$$

and may be regarded as one basis vector. (The factor 2 arises from the fact that both eg. $z_1 z_2 z_3^2$ and $z_2 z_1 z_3^2$ occur in the sum.) Since $\psi_{(L=N)} = \mathcal{S}[f_U]$ (\mathcal{S} denoting symmetrization), it is therefore straightforward to determine the representation of $\psi_{(L=N)}$ in the symmetric basis by symmetrizing each term of f_U individually.

Since overlap integrals between polynomial terms with non-matching powers are zero, our symmetric basis is orthogonal. This means that if both $\psi_{(L=N)}$ and $\psi_{(L=N)}^{ex}$ are normalized and represented in a normalized symmetric basis, then the overlap may be computed as a simple dot product. This also implies that all calculated overlap values are exact (up to numerical precision). To express $\psi_{(L=N)}^{ex}$ in this basis, let us rephrase the product (14) as the sum

$$\psi_{(L=N)}^{ex} = \sum_{i=0}^N \frac{1}{(-N)^i} s_1^i \cdot s_{N-i}, \quad (25)$$

where $s_i = s_i(z_1, \dots, z_N)$ are the fundamental symmetric polynomials of degree i ,

$$s_i(z_1, \dots, z_N) = \mathcal{S}[z_1 z_2 \dots z_i]. \quad (26)$$

The products $s_N, s_{N-1}, \dots, s_1^N$ appearing in this sum form another, non-orthogonal, basis. Through a change

of basis we can therefore convert $\psi_{(L=N)}^{ex}$ to the orthogonal symmetrized basis, enabling us to compute the overlaps.

The method described in this section may be straightforwardly modified to other values of L . In the next section we illustrate how it may be used to shed light on some yrast states at even lower angular momenta.

BELOW THE SINGLE VORTEX, $L < N$

Exact ground state wave functions are known not only for the single vortex (14) but for all angular momentum states $2 \leq L \leq N$. As was shown some years ago[14, 15], they are simply given by fundamental symmetric polynomials $s_L(\tilde{z}_i)$ where $\tilde{z}_i = z_i - Z$ and Z is again the center-of-mass coordinate:

$$\psi_L^{ex} = \sum_{p_1 < p_2 < \dots < p_L} (z_{p_1} - Z)(z_{p_2} - Z) \cdots (z_{p_L} - Z). \quad (27)$$

For example, $\psi_{L=2} = \mathcal{S}[(z_1 - Z)(z_2 - Z)]$, with \mathcal{S} denoting symmetrization over all particle coordinates. Note that these states are *translation invariant* (TI), i.e. invariant under a simultaneous constant shift $z_i \rightarrow z_i + a$ of all the coordinates[11, 20]. We see that the single vortex (14) is merely a special case of this series of wave functions. (At $L = 0$ the polynomial part of the exact ground state is trivially a constant as all particles are in the $l = 0$ state, while the only possible way of constructing the ground state at $L = 1$ is $\psi_{L=1} = \sum_i z_i$, which corresponds to a center of mass excitation of the $L = 0$ state.)

As discussed in Refs. [11, 20], the space of TI polynomials for $2 \leq L \leq N$ is spanned by the basis states

$$|k_2 k_3 \dots k_N\rangle = s_2^{k_2}(\tilde{z}_i) s_3^{k_3}(\tilde{z}_i) \cdots s_N^{k_N}(\tilde{z}_i) \quad (28)$$

where $L = \sum_{n=2}^N n k_n$. We thus see that the states (27) are special cases of such basis states, with $k_L = 1$, all other $k_n = 0$. In anticipation of the discussion below, note that for $L = 2, 3$, they are the *only* possible basis states, and thus it is obvious that (27) has to be exact for $L = 2, 3$. (For larger L this is no longer the case; for example, the basis for $L = 4$ consists of $|20000\dots\rangle$ as well as $|00100\dots\rangle$.)

Given these results, let us discuss some mathematical peculiarities of the CF construction in the regime $L < N$. First of all, the CF construction immediately reproduces the exact wave function for $L = 0$: In order to cancel all the angular momentum of the Jastrow factor, the CF Slater determinant with lowest possible (quasi) Landau level energy is uniquely given by occupying the CF Lan-

dau level states $\eta_{n,-n}$ for $n = 0$ through $N - 1$,

$$f_S = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \bar{z}_1 & \bar{z}_2 & \dots & \bar{z}_N \\ \bar{z}_1^2 & \bar{z}_2^2 & \dots & \bar{z}_N^2 \\ \dots & \dots & \dots & \dots \\ \bar{z}_1^{N-1} & \bar{z}_2^{N-1} & \dots & \bar{z}_N^{N-1} \end{vmatrix} = \prod_{i < j} (\bar{z}_i - \bar{z}_j). \quad (29)$$

After projection, this will give a Jastrow factor of derivatives acting on the corresponding Jastrow factor of z 's, thus resulting in a constant. This can be made explicit using the notation of the previous section. Proceeding exactly as in equations (16)-(22), again taking $N = 4$, we find that $\psi_{L=0} = \sum_{\pi} f_U(\pi(1), \pi(2), \pi(3), \pi(4))$ with

$$\begin{aligned} f_U &= \sum_{\sigma} \varepsilon(\sigma) \partial_1^0 \partial_2^1 \partial_3^2 \partial_4^3 \times z_1^{\sigma(1)} z_2^{\sigma(2)} z_3^{\sigma(3)} z_4^{\sigma(4)} \\ &= \sum_{\sigma} \varepsilon(\sigma) z_1^{\sigma(1)} \cdot \sigma(2) z_2^{\sigma(2)-1} \cdot \sigma(3)(\sigma(3)-1) z_3^{\sigma(3)-2} \\ &\quad \cdot \sigma(4)(\sigma(4)-1)(\sigma(4)-2) z_4^{\sigma(4)-3} \end{aligned} \quad (30)$$

where σ now denotes all permutations over $(0,1,2,3)$. This is identified with the determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ z_1 & 1 & 0 & 0 \\ z_1^2 & 2z_2 & 2 & 0 \\ z_1^3 & 3z_2^2 & 6z_3 & 6 \end{vmatrix} = 12 \quad (32)$$

which obviously gives the correct wave function (up to normalization) for $L = 0$ (all bosons in the $l = 0$ state).

The CF construction reproduces the *exact* ground state wave functions for $L = 2$ and 3 as well. In contrast to the case $L = N$, these identities are not asymptotic – they are exact for *all* N . This is not surprising, for the following reason[11]: For these angular momenta, one can construct *compact* CF wave functions which are thus known to be translation invariant (TI)[21]. On the other hand, as discussed above, for these lowest angular momenta there only exists one basis state in the space of TI polynomials. Therefore, the CF compact state has to correspond to this basis state, which is the exact wave function.

What *is* somewhat surprising, however, is how this comes about mathematically. Consider again the $L = 0$ state discussed above. From this we can construct compact $L = 2$ states by moving *any* of the particles (except the one at $(0,0)$) from the CF Landau level state $(n, -n)$ to $(n-1, -n+2)$. There are thus $N - 1$ different ways of constructing the $L = 2$ CF Slater determinant which are all degenerate in CF kinetic energy. In mathematical terms, this corresponds to replacing one of the states $\eta_{n,-n} = \partial^n$ by $z\partial^{n-1}$ (see (7)). It thus looks as if there are $N - 1$ different CF candidate wave functions for the $L = 2$ state, while according to the above arguments, we would expect the CF wave function to be unique and exact! The solution to this apparent paradox is that *while*

the $N - 1$ possible Slater determinants are certainly not equal to each other, they all result in the same polynomial after acting on the Jastrow factor. To see how this comes about, consider again the case of four particles. The first possible Slater determinant is

$$f_S = \begin{vmatrix} z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \\ \partial_1^2 & \partial_2^2 & \partial_3^2 & \partial_4^2 \\ \partial_1^3 & \partial_2^3 & \partial_3^3 & \partial_4^3 \end{vmatrix} \quad (33)$$

resulting in

$$f_U = \sum_{\sigma} \varepsilon(\sigma) z_1 z_2^0 \partial_3^2 \partial_4^3 \times z_1^{\sigma(1)} z_2^{\sigma(2)} z_3^{\sigma(3)} z_4^{\sigma(4)} \quad (34)$$

$$\begin{aligned} &= \sum_{\sigma} \varepsilon(\sigma) z_1^{\sigma(1)+1} \cdot z_2^{\sigma(2)} \cdot \sigma(3)(\sigma(3)-1) z_3^{\sigma(3)-2} \\ &\quad \cdot \sigma(4)(\sigma(4)-1)(\sigma(4)-2) z_4^{\sigma(4)-3} \end{aligned} \quad (35)$$

which equals the determinant

$$f_U = \begin{vmatrix} z_1 & 1 & 0 & 0 \\ z_1^2 & z_2 & 0 & 0 \\ z_1^3 & z_2^2 & 2 & 0 \\ z_1^4 & z_2^3 & 6z_3 & 6 \end{vmatrix} = 12(z_1 z_2 - z_1^2). \quad (36)$$

Proceeding in the same way for the other two possibilities gives $f_U = 12(z_3^2 - z_2 z_3)$ and $f_U = 12(z_4^2 - z_3 z_4)$, respectively. Thus, since the *full* wave function is constructed from f_U by summing over all permutations of the indices $(1,2,3,4)$, the final, symmetrized sum will be the same for all these constructions.

Exactly the same thing happens for $L = 3$. There are $N - 2$ ways of constructing the Slater determinant, obtained from the $L = 0$ one by letting any $\partial^n \rightarrow z\partial^{n-2}$. Again, these different constructions simply correspond to picking out different parts of the final, symmetrized polynomial.

Of course, similar things may happen at other angular momenta as well. For example, there are two ways of constructing the Slater determinant for $L = N - 1$. Again, these result in the same polynomial, though in this case the CF construction does not give the exact wave function even in the large N limit.

The fact that the wave functions resulting from the CF construction for $L = 2, 3$ are equal to the very simple and compact expressions in (27), again provides non-trivial and potentially useful mathematical identities.

SUMMARY AND OUTLOOK

To summarize, we have shown that the CF construction for bosons at low angular momenta may be taken to far higher particle numbers than previously by handling the derivatives of the CF Slater determinant in a new way. In particular, we used this to show that for

the single vortex, $L = N$, the CF construction becomes exact in the high N limit, leading to a non-trivial and potentially useful mathematical identity. For other L , the CF construction does, in general, *not* produce the exact wave functions. We have argued why it is expected to give the exact ground state for $L = 2, 3$ (all N). However, work remains to get a better understanding of what makes $L = N$ special. Moreover, we used our methods to show examples of other mathematical peculiarities of the CF construction in the regime $L < N$. Our hope is that the results of this paper may provide new insight into properties of CF wave functions in the few-vortex regime beyond $L = N$, where no exact, analytical wave functions are known. For example, a preliminary study of the case $L = 2N$ suggests that while the number of possible CF Slater determinants increases linearly with N , these may reduce to just two different wave functions; finding the ground state as a linear combination of these would then just be a one-parameter problem. There is thus hope that one may construct rather simple analytical trial wave functions in the regime $L \sim N$, which may directly reveal the vortex structures of these states. The present paper indicates that in this regime, the CF construction may do far better than previously assumed in the literature.

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